

CRITERIA FOR THE ABSENCE AND EXISTENCE OF ARBITRAGE IN MULTI-DIMENSIONAL DIFFUSION MARKETS

DAVID CRIENS

ABSTRACT. In this article, we study the set of equivalent (local) martingale measures for financial markets driven by multi-dimensional diffusions. We give conditions for the existence of equivalent (local) martingale measures in terms of existence and uniqueness properties of martingale problems. Based on these we derive deterministic criteria for the existence and non-existence of equivalent (local) martingale measures. As an application, we construct a financial market in which the number of risky assets determines the absence of arbitrage and equals the number of sources of risk.

1. INTRODUCTION

The question when a financial model is free of arbitrage is typically ask for each financial model individually. Our goal is to provide a systematic discussion for classes of models driven by multi-dimensional diffusions.

We explain our setting in more detail. Let X be the coordinate process on the path-space Ω of continuous functions $[0, \infty) \rightarrow \mathbb{R}^d$ and let \mathcal{F} be the σ -field generated by X . The real-world measure P of our financial market is a probability measure on (Ω, \mathcal{F}) such that X is a diffusion parameterized by a drift coefficient b and a diffusion coefficient a . Below, we will formally introduce P as a solution to a martingale problem. Our financial market consists of $m \leq d$ risky assets. Each of them is modeled as a discounted price process $(S_t^i)_{t \in [0, T]}$ which is assumed to be the stochastic exponential

$$(1.1) \quad dS_t^i = S_t^i \langle e_i, dX_t \rangle$$

with deterministic initial value $S_0^i > 0$. Here, $T \in (0, \infty)$ is a finite time horizon, e_i is the i -th unit vector and $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. Since $m < d$ is possible, our setting includes incomplete markets and stochastic volatility models.

For this financial market the classical concepts of no-arbitrage are the notion of *no free lunch with vanishing risk (NFLVR)* as defined by Delbaen and Schachermayer [4, 5] and the notion *no generalized arbitrage (NGA)* as defined by Cherny [2] and Yan [25]. The difference between (NFLVR) and (NGA) is captured by the concept of a *financial bubble* in the sense of Cox and Hobson [3]. More precisely, a financial bubble exists if (NFLVR) holds while (NGA) fails. In diffusion markets it is well-known that (NFLVR) is equivalent to the existence of an *equivalent local martingale measure (ELMM)*, cf. [4, 5], and that (NGA) is equivalent to the existence of an *equivalent martingale measure (EMM)*, cf. [2]. Consequently, a financial bubble exists if there is an ELMM but no EMM.

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D. Crien - Technical University of Munich, Department of Mathematics, Germany, david.criens@tum.de.

The contributions of this article are twofold. First, we identify solutions to martingale problems as E(L)MMs and derive conditions for the existence of E(L)MMs in terms of existence and uniqueness properties of martingale problems. Second, we give deterministic criteria for the existence and non-existence of E(L)MMs. These criteria only depend on the parameters b and a , are typically weaker than classical Novikov-type conditions and easy to verify.

Let us comment on existing literature. The absence of arbitrage for a class of diffusion models was for instance studied by Lyasoff [15], Delbaen and Shirakawa [6], and Mijatović and Urusov [17]. Lyasoff works in a market driven by an Itô process. He shows that (NFLVR) is determined by the equivalence of two probability measures, one of them being the Wiener measure. The spirit of the work of Delbaen and Shirakawa and Mijatović and Urusov is close to ours. Their goal is to derive deterministic criteria which are suitable for applications. However, they work in a one-dimensional setting, while we are particularly interested in multi-dimensional cases. We illustrate that multi-dimensional settings differ significantly from the one-dimensional case by constructing a market which allows for arbitrage opportunities only if it features more than three risky assets. This is particularly surprising when noting that the sources of risk in this market have the same dimension.

This article is structured as follows. In Section 2 we recall the definitions of martingale problems and E(L)MMs. Our main results are given in Section 3. More precisely, in Section 3.1 we characterize the set of E(L)MMs in terms martingale problems and in Section 3.2 we prove deterministic conditions for the existence and non-existence of E(L)MMs. Finally, we present an application in Section 4.

2. THE PROBABILISTIC SETUP

Let Ω be the space of continuous function $[0, \infty) \rightarrow \mathbb{R}^d$ for a fixed $d \in \mathbb{N}$. The coordinate process X on Ω is given by $X_t(\omega) = \omega(t)$. We define by $\mathcal{F} \equiv \sigma(X_t, t \in [0, \infty))$ a σ -field on Ω . It is well-known that \mathcal{F} is the Borel σ -field on Ω in the case where Ω is equipped with the local uniform topology. We set $\mathcal{F}_t \equiv \sigma(X_s, s \in [0, t])$ for $t \in [0, \infty)$, $\mathbf{F} \equiv (\mathcal{F}_t)_{t \in [0, \infty)}$ and $\mathbf{F}^+ \equiv (\mathcal{F}_{t+})_{t \in [0, \infty)}$. Our financial model will be parameterized by a tuple (b, a) , where

- (i) $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a locally bounded Borel function.
- (ii) $a: \mathbb{R}^d \rightarrow \mathbb{S}^d$ is a locally bounded Borel function and \mathbb{S}^d is the set of all positive definite $d \times d$ matrices.

The parameter b corresponds to the drift and the parameter a corresponds to the diffusion part.

The probabilistic concept underlying our model is the martingale problem as introduced by Stroock and Varadhan [23]. Let $C_c^2(\mathbb{R}^d)$ be the set of twice continuously differentiable functions $\mathbb{R}^d \rightarrow \mathbb{R}$ with compact support.

Definition 2.1. *Let $x \in \mathbb{R}^d$. We call a probability measure P on (Ω, \mathcal{F}) a solution to the martingale problem (MP) (b, a, x) , if $P(X_0 = x) = 1$ and for all $f \in C_c^2(\mathbb{R}^d)$ the process*

$$(2.1) \quad M_t^f \equiv f(X_t) - f(x) - \int_0^t \left(\langle b(X_s), \nabla f(X_s) \rangle + \frac{1}{2} \text{trace}(a(X_s) \nabla^2 f(X_s)) \right) ds$$

is an (\mathbf{F}, P) -martingale. Here, ∇f denotes the gradient of f and ∇^2 denotes the Hessian matrix of f . We call the MP (b, a) well-posed if for all $x \in \mathbb{R}^d$ there exists a unique solution to the MP (b, a, x) . The set of solutions is denoted by $\mathcal{M}(b, a, x)$.

Remark 2.2. (i) Sometimes it is more convenient to work with the right-continuous filtration \mathbf{F}^+ instead of the natural filtration \mathbf{F} . For the concept

of MPs it does not play any role which filtration we consider. Indeed, an F -adapted right-continuous process is an F -martingale if and only if it is an F^+ -martingale. The implication \implies is implied by the backward martingale convergence theorem and the implication \impliedby is a consequence of the tower rule. Note that for any continuous local martingale Y the sequence $(\inf(t \in [0, \infty): \|Y_t\| \geq n))_{n \in \mathbb{N}}$ is a localizing sequence. In particular, it is a sequence of F -stopping times whenever Y is F -adapted. Hence, any F -adapted process is a continuous local F -martingale if and only if it is a continuous local F^+ -martingale.

- (ii) Let us also comment on the semimartingale property under F and F^+ and on the measurability of stochastic integrals. Note that the predictable σ -field coincides for the filtrations F and F^+ , cf., e.g., [16, Corollary 3.5]. Therefore, any F -adapted process is a continuous F -semimartingale if and only if it is a continuous F^+ -semimartingale. Typically, stochastic integration is introduced under the hypothesis that the filtration is augmented. However, since our proofs rely on measure changes, we do not want to impose this assumption. Stroock and Varadhan [23] defined an F -progressively measurable stochastic integral process $\int_0^\cdot \langle \theta_u, d\xi_u \rangle$ for an F -progressively measurable integrand θ and an F -progressively measurable Itô process ξ as integrator. This integral process has the same well-known properties as the one defined for the augmented filtration. For a detailed discussion we refer to [23, Section 4.3].
- (iii) We remark that MPs can be reformulated in terms of classical semimartingale theory. More precisely, a probability measure P on (Ω, \mathcal{F}) solves the MP (b, a, x) if and only if $P(X_0 = x) = 1$ and the coordinate process X is a continuous (F, P) -semimartingale with decomposition

$$X = x + \int_0^\cdot b(X_s) ds + X^c,$$

where X^c is a continuous local (F, P) -martingale with quadratic variation process $\int_0^\cdot a(X_s) ds$ and $X_0^c = 0$. This decomposition is unique up to a P -null set. For a proof we refer to the proofs of [10, Theorem 13.55] and [13, Proposition 5.4.11]. This equivalence is important when we apply Girsanov's theorem.

Next, we also introduce a *stopped* martingale problem.

Definition 2.3. For an F -stopping time τ we call a probability measure P on (Ω, \mathcal{F}) a solution to the (stopped) MP (b, a, x, τ) if the stopped process $M_{\cdot \wedge \tau}^f$ is an (F, P) -martingale for all $f \in C_c^2(\mathbb{R}^d)$. We denote the set of solutions by $\mathcal{M}(b, a, x, \tau)$.

We note that MPs have the following *local uniqueness* property.

Lemma 2.4. Suppose that the MP (b, a) is well-posed. Let τ and ρ be F -stopping times and $x \in \mathbb{R}^d$. For all $P \in \mathcal{M}(b, a, x, \tau)$ and $Q \in \mathcal{M}(b, a, x, \rho)$ it holds that $P = Q$ on $\mathcal{F}_{\tau \wedge \rho}$.

Proof: The hypothesis that the MP (b, a) is well-posed together with [23, Exercise 6.7.4] and Remark 2.2 yield that the assumptions of [11, Lemma IX.4.4] are satisfied. This lemma implies our claim. \square

We now turn to our financial setting. Let $T \in (0, \infty)$ be a finite time horizon. In this article we suppose that P is a solution to the MP (b, a, x_0, T) with $\|x_0\| = 1$. Our market is supposed to support $m \leq d$ risky assets. Each of these will be

represented by a discounted stock-price processes S^i . Formally, for $i = 1, \dots, m$, we set

$$S^i \equiv S_0^i \exp \left(\langle e_i, X_{\cdot \wedge T} \rangle - \langle e_i, x_0 \rangle - \frac{1}{2} \int_0^{\cdot \wedge T} \langle a(X_s) e_i, e_i \rangle ds \right)$$

for a deterministic initial value $S_0^i > 0$. This definition coincides with (1.1).

Definition 2.5. *We call a probability measure Q on (Ω, \mathcal{F}) an equivalent (local) martingale measure (E(L)MM), if $Q \sim P$ on \mathcal{F}_T and S is a (local) (\mathbf{F}, Q) -martingale. We denote the set of E(L)MMs by $\mathcal{M}_{(l)}$.*

Thanks to the seminal work of Delbaen and Schachermeyer [4], the existence of an ELMM is equivalent to the (NFLVR) condition. Moreover, Cherny [2] showed that (NGA) is equivalent to the existence of an EMM.

In the following section we present necessary and sufficient conditions for the existence of an E(L)MM in our multi-dimensional diffusion market.

3. MAIN RESULTS

Stochastic exponentials of local martingales are itself local martingales. Hence, any solution to the MP $(0, a, x_0, T)$ is a good candidate for an E(L)MM. As we will see in Section 3.1 below, under some uniqueness assumptions, in the complete setting it is even the only candidate for an E(L)MM. In the incomplete case, a mild drift condition allows us to identify a set of solutions as candidates for E(L)MMs. Consequently, the existence and non-existence of one or even infinitely many E(L)MMs boils down to the question when a MP is well-posed. Using this observation we can give deterministic conditions which only depend on b and a . We formulate them in Section 3.2 below.

Before we start this program we introduce a *minimal* condition. It is based on the following well-known drift condition, cf., e.g., [20]. For sake of completeness we give a full proof.

Lemma 3.1. *If $\mathcal{M}_l \neq \emptyset$, then there exists an \mathbf{F} -predictable process c such that for all $i = 1, \dots, m$ and P -a.s. for all $t \in [0, \infty)$*

$$(3.1) \quad \int_0^{t \wedge T} \langle e_i, b(X_s) - a(X_s)c_s \rangle ds = 0.$$

Proof: Let $i \in \{1, \dots, m\}$ be fixed and take $Q \in \mathcal{M}_l$. We denote by \mathcal{L} the stochastic logarithm. Recall that the predictable σ -field coincides for the filtrations \mathbf{F} and \mathbf{F}^+ . By Girsanov's theorem [11, Theorem III.3.24] there exists an \mathbf{F} -predictable process c such that Q -a.s.

$$\mathcal{L}(S^i) = \log(S_0) + \int_0^{\cdot \wedge T} \langle e_i, b(X_s) - a(X_s)c_s \rangle ds + \text{local } (\mathbf{F}, Q)\text{-martingale}.$$

Recall that the stochastic logarithm of a continuous local martingale is also a continuous local martingale. Hence, since the process S^i is a local (\mathbf{F}, Q) -martingale, also $\mathcal{L}(S^i)$ is a local (\mathbf{F}, Q) -martingale. Now, the claim of our lemma follows from the fact that continuous local martingales of bounded variation are constant up to a null set. \square

We introduce the \mathbf{F} -stopping time

$$\tau_n \equiv \inf(t \in [0, \infty) : \|X_t\| \geq n) \wedge n.$$

Let us formulate the

Condition M. 1. *There exists an F -predictable process c and a locally bounded Borel function $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^{d-m}$ such that for all $i = 1, \dots, m$ the equality (3.1) holds P -a.s. for all $t \in [0, \infty)$, and for all $i = m+1, \dots, d$ it holds P -a.s. for all $t \in [0, \infty)$ that*

$$\int_0^{t \wedge T} \langle e_i, b(X_s) - a(X_s)c_s \rangle ds = \int_0^{t \wedge T} \langle e_{i-m}, \mu(X_s) \rangle ds.$$

Moreover, for all $n \in \mathbb{N}$

$$(3.2) \quad E^P \left[\exp \left(\frac{1}{2} \int_0^{T \wedge \tau_n} \langle a(X_s)c_s, c_s \rangle ds \right) \right] < \infty.$$

Remark 3.2. In the case $m = d$, Condition M.1 boils down to the necessary drift condition of Lemma 3.1 together with the local Novikov condition (3.2), which is for instance satisfied if $c_s(\omega) = c(\omega(s))$ and $c: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally bounded. In the case $m < d$, we impose the additional assumption that we may find a candidate E(L)MM which potentially solves a MP. In many cases this assumption holds. For instance, suppose that for all $x \in \mathbb{R}^d$ the matrix $a(x)$ is invertible with inverse $a^{-1}(x)$ and that $a^{-1}b: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally bounded. Then, we can set $c_t(\omega) \equiv -a^{-1}(\omega(t))b(\omega(t)) + (0, \mu(\omega(t)))$ for any locally bounded Borel function $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^{d-m}$. We think that Condition M.1 is close to being necessary for the existence of an E(L)MM.

3.1. E(L)MMs and MPs. Condition M.1 guarantees that the solutions of the MP $((0, \mu), a, x_0, T)$ are candidate ELMMs. The following theorem states that if the MP $((0, \mu), a)$ is well-posed, then the (unique) candidate is an ELMM.

Theorem 3.3. *Suppose that Condition M.1 holds and that the MP $((0, \mu), a)$ is well-posed. It holds that*

$$(3.3) \quad \mathcal{M}((0, \mu), a, x_0, T) \subseteq \mathcal{M}_l.$$

Proof: We set

$$X^c \equiv X - \int_0^{\cdot \wedge T} b(X_s) ds,$$

and note that X^c is a continuous local (F, P) -martingale with quadratic variation process $\int_0^{\cdot \wedge T} a(X_s) ds$. Since (3.2) holds, Novikov's condition yields that

$$Z_{\cdot \wedge T \wedge \tau_n} \equiv \exp \left(\int_0^{\cdot \wedge T \wedge \tau_n} \langle c_s, dX_s^c \rangle - \frac{1}{2} \int_0^{\cdot \wedge T \wedge \tau_n} \langle a(X_s)c_s, c_s \rangle ds \right)$$

is an (F, P) -martingale. Define a probability measure Q^n on (Ω, \mathcal{F}) by the Radon-Nikodym derivative $dQ^n = Z_{T \wedge \tau_n} dP$. Denote $Q_{x_0} \in \mathcal{M}((0, \mu), a, x_0, T)$.

Lemma 3.4. $Q_{x_0} = Q^n$ on $\mathcal{F}_{T \wedge \tau_n}$.

Proof: By Girsanov's theorem [11, Theorem III.3.24], Remark 2.2 and the assumptions on c and μ given by Condition M.1, $Q^n \in \mathcal{M}((0, \mu), a, x_0, T \wedge \tau_n)$. Now, Lemma 2.4 implies the claim. \square

For all $A \in \mathcal{F}_T$, by the monotone convergence theorem, we have

$$(3.4) \quad \begin{aligned} Q_{x_0}(A) &= \lim_{n \rightarrow \infty} Q_{x_0}(A \cap \{\tau_n > T\}) \\ &= \lim_{n \rightarrow \infty} Q^n(A \cap \{\tau_n > T\}) \\ &= \lim_{n \rightarrow \infty} E^P [Z_{T \wedge \tau_n} \mathbf{1}_{A \cap \{\tau_n > T\}}] \\ &= \lim_{n \rightarrow \infty} E^P [Z_T \mathbf{1}_{A \cap \{\tau_n > T\}}] \\ &= E^P [Z_T \mathbf{1}_A]. \end{aligned}$$

We deduce from $P(Z_T > 0) = 1$ that $Q_{x_0} \sim P$ on \mathcal{F}_T . Since $\langle e_i, X_{\cdot \wedge T} \rangle$ is a local (\mathbb{F}, Q_{x_0}) -martingale by the definition of Q_{x_0} , so is its stochastic exponential S^i . Therefore, (3.3) is proven. \square

We stress that Theorem 3.3 also gives a condition when the market is incomplete, i.e. more than one ELMM exists.

Let us consider a complete market, i.e. $m = d$. In this case, all ELMMs are solutions to the MP $(0, a, x_0, T)$. In particular, this holds without any further assumptions.

Theorem 3.5. *If $m = d$, then*

$$\mathcal{M}_l \subseteq \mathcal{M}(0, a, x_0, T).$$

Proof: Suppose that $Q \in \mathcal{M}_l$. Then S is a continuous local (\mathbb{F}, Q) -martingale. Moreover, $X_{\cdot \wedge T} = \mathcal{L}(S_0^{-1}S)$ is a continuous local (\mathbb{F}, P) -martingale. Since, by Girsanov's theorem [11, Theorem III.3.24], the quadratic variation of $X_{\cdot \wedge T}$ is the same for P and Q , it follows that $Q \in \mathcal{M}(0, a, x_0, T)$. \square

Combining the Theorems 3.3 and 3.5 we obtain the following corollary which gives a precise description of the set of ELMMs in the complete case. The uniqueness is a consequence of Lemma 2.4.

Corollary 3.6. *Suppose that Condition M.1 holds and that the MP $(0, a)$ is well-posed. If $m = d$, then*

$$(3.5) \quad \mathcal{M}(0, a, x_0, T) = \mathcal{M}_l.$$

In particular, $|\mathcal{M}_l|_{\mathcal{F}_T} = 1$, i.e. all ELMMs coincide on \mathcal{F}_T .

Next, we study the set of EMMs. Since each EMM has to be an ELMM, we only have to find sufficient conditions for an ELMM to be an EMM. In the proof of Theorem 3.3 we used a local change of measure to show that the solutions to the MPs (b, a, x_0) and $(0, a, x_0)$ are equivalent on \mathcal{F}_T . As a byproduct, the local density process, called Z in the proof of Theorem 3.3, is a martingale. Hence, to derive conditions for the martingale property of S^i , we only have to repeat this proof with Z replaced by S^i . In view of Theorem 3.8 below, we think that our conditions are close to being necessary.

Theorem 3.7. *Suppose that Condition M.1 holds and that the MP $((0, \mu), a)$ is well-posed. Moreover, assume that the MP $((0, \mu) + ae_i, a)$ is well-posed for all $i = 1, \dots, m$. Then*

$$\mathcal{M}((0, \mu), a, x_0, T) \subseteq \mathcal{M}.$$

In particular, if $m = d$, then

$$(3.6) \quad \mathcal{M}(0, a, x_0, T) = \mathcal{M}$$

and $|\mathcal{M}|_{\mathcal{F}_T} = 1$, i.e. all EMM coincide on \mathcal{F}_T .

Proof: It follows from Theorem 3.3 that $Q_{x_0} \in \mathcal{M}_l$. Hence, it only remains to show that our additional assumptions imply that for all $i = 1, \dots, m$ the process S^i is not only a local (\mathbb{F}, Q_{x_0}) -martingale, but a true (\mathbb{F}, Q_{x_0}) -martingale. Since non-negative local martingales are supermartingales, this is the case if and only if $E^{Q_{x_0}}[S_T^i] = S_0$. We proceed along the lines of the proof to Theorem 3.3. In the following we fix $i \in \{1, \dots, m\}$. Due to the assumption that a is locally bounded, we have

$$\int_0^{T \wedge \tau_n} \langle a(X_s) e_i, e_i \rangle ds \leq T \sup_{\|x\| \leq n} \|a(x)\| < \infty.$$

Hence, Novikov's condition implies that $S_{\cdot \wedge \tau_n}^i$ is an (F, Q_{x_0}) -martingale. Define a probability measure $Q_{x_0}^n$ on (Ω, \mathcal{F}) by the Radon-Nikodym derivative $dQ_{x_0}^n = S_{\tau_n \wedge T}^i / S_0^i dQ_{x_0}$. Similarly to the proof of Lemma 3.4 we deduce from Lemma 2.4 that

$$Q_{x_0}^* = Q_{x_0}^n \text{ on } \mathcal{F}_{T \wedge \tau_n},$$

where $Q_{x_0}^* \in \mathcal{M}((0, \mu) + ae_i, a, x_0)$. Hence, as in (3.4), we obtain

$$E^{Q_{x_0}}[S_T^i] = \lim_{n \rightarrow \infty} E^{Q_{x_0}^n}[S_0 \mathbf{1}_{\{\tau_n > T\}}] = \lim_{n \rightarrow \infty} E^{Q_{x_0}^*}[S_0 \mathbf{1}_{\{\tau_n > T\}}] = S_0.$$

Therefore, S^i is an (F, Q_{x_0}) -martingale and we conclude $Q_{x_0} \in \mathcal{M}$. The equality (3.6) follows from the inclusion $\mathcal{M} \subseteq \mathcal{M}_l$ and Theorem 3.5. The uniqueness is a consequence of Lemma 2.4. \square

In the following theorem we give a necessary condition for the existence of an EMM.

Theorem 3.8. *If $m = d$ and $\mathcal{M}(ae_i, a, x_0, T) = \emptyset$ for at least one $i \in \{1, \dots, m\}$, then $\mathcal{M} = \emptyset$.*

Proof: Suppose that $Q \in \mathcal{M}$. Since $\mathcal{M} \subseteq \mathcal{M}_l$, Theorem 3.5 yields that $Q \in \mathcal{M}(0, a, x_0, T)$. In particular, S^i is an (F, Q) -martingale. We can define a probability measure Q^i on (Ω, \mathcal{F}) by the Radon-Nikodym derivative $dQ^i = S_T^i / S_0^i dQ$. Now, Girsanov's theorem [11, Theorem III.3.24] together with Remark 2.2 yields that Q^i solves the MP (ae_i, a, x_0, T) . This contradiction implies the claim. \square

3.2. Deterministic Conditions for the Existence and Non-Existence of E(L)MMs. This section is split into two parts. In the first, we give deterministic conditions for the existence of E(L)MMs. We base them on existence and uniqueness results for MPs as given by Stroock and Varadhan [23]. In the second, we derive deterministic conditions for the non-existence of E(L)MMs. This is more difficult than giving conditions for the existence, since classical non-existence results for MPs are designed for the infinite time horizon and we work with a finite time horizon.

Let us start with classical uniqueness conditions.

Condition U.1. *The function a is continuous and for all $x \in \mathbb{R}^d$*

$$\inf_{\|\theta\|=1} \langle a(x)\theta, \theta \rangle > 0.$$

Condition U.2. *Each entry of σ is twice continuously differentiable and each entry of μ is once continuously differentiable.*

Both of these conditions imply the following uniqueness result.

Lemma 3.9. *Suppose that one of the Conditions U.1 or U.2 holds. Then, for all $x \in \mathbb{R}^d$ and $i = 1, \dots, m$, the MPs $(0, a, x)$, $((0, \mu), a, x)$ and $((0, \mu) + ae_i, a, x)$ have at most one solution.*

Proof: In the case where Condition U.1 holds, [23, Theorem 10.1.3] implies the claim.

Suppose that Condition U.2 holds. We note that a , being twice continuously differentiable, has a locally Lipschitz continuous root, cf. [9, Proposition IV.6.2]. Moreover, the functions $(0, \mu)$ and $(0, \mu) + ae_i$, being continuously differentiable, are locally Lipschitz continuous. Now, we only need to use that for SDEs local Lipschitz continuous coefficients imply pathwise uniqueness which itself implies uniqueness in law, and that SDEs and MPs have a one-to-one correspondence. More precisely, [13, Theorem 5.2.5, Proposition 5.3.20, Corollary 5.4.8] together with similar arguments as used in the proof of [13, Proposition 5.4.11] yield the claim. \square

3.2.1. *Deterministic Conditions for the Existence of $E(L)$ MMs.* The following condition can be seen as a (partial) multi-dimensional Feller test for explosion. It goes back to Khas'minskii [14] who stated it without proof. Providing an intuition, the condition is based on a radial comparison with a one-dimensional diffusion.

Condition EL.1. *There exists an $r > 0$ and continuous functions $A: [r, \infty) \rightarrow (0, \infty)$ and $B: [r, \infty) \rightarrow (0, \infty)$ such that for $\rho \geq \sqrt{2r}$ and $|x| = \rho$,*

$$(3.7) \quad \begin{aligned} A\left(\frac{\rho^2}{2}\right) &\geq \langle a(x)x, x \rangle, \\ \langle a(x)x, x \rangle B\left(\frac{\rho^2}{2}\right) &\geq \text{trace } a(x) + 2\langle x, (0, \mu)(x) \rangle, \\ \int_r^\infty \frac{\int_r^z \frac{C(\sigma)}{A(\sigma)} d\sigma}{C(z)} dz &= \infty, \end{aligned}$$

where

$$(3.8) \quad C(z) \equiv \exp\left(\int_r^z B(\sigma) d\sigma\right).$$

We also give a second condition which is easier to verify. In contrast to Condition EL.1, the dimension is not taken into consideration.

Condition EL.2. *For all $x \in \mathbb{R}^d$ there exists a constant $C > 0$ such that*

$$(3.9) \quad \|a(x)\| \leq C(1 + \|x\|^2),$$

$$(3.10) \quad \langle x, (0, \mu)(x) \rangle \leq C(1 + \|x\|^2).$$

Now, we obtain the following

Corollary 3.10. *Suppose the following:*

- (i) *Condition M.1 holds.*
- (ii) *One of the Conditions U.1 and U.2 holds.*
- (iii) *One of the Conditions EL.1 and Condition EL.2 holds.*

Then $\emptyset \neq \mathcal{M}((0, \mu), a, x_0, T) \subseteq \mathcal{M}_l$. Moreover, if $m = d$, then $\mathcal{M}_l = \mathcal{M}(0, a, x_0, T)$ and $|\mathcal{M}_l|_{\mathcal{F}_T} = 1$.

Proof: Thanks to Theorem 3.3 and Corollary 3.6 it suffices to show that the MP $((0, \mu), a)$ is well-posed. We want to apply [23, Theorem 10.2.2, Theorem 10.2.3]. It suffices to check that either of the Conditions U.1 and U.2 imply the assumptions of [23, Corollary 10.1.2]. More precisely, that means we have to show that for all $n \in \mathbb{N}$ there exist bounded Borel functions a_n and μ_n such that $a_n = a$ and $\mu_n = (0, \mu)$ on $\{x \in \mathbb{R}^d: \|x\| \leq n\}$ and the MP (μ_n, a_n) is well-posed. In the case where Condition U.1 holds, these assumptions hold as discussed in [23, pp. 250]. Suppose that Condition U.2 holds. Let $\Phi_n \in C_c^2(\mathbb{R}^d)$ be such that $0 \leq \Phi_n \leq 1$ and $\Phi_n = 1$ on $\{x \in \mathbb{R}^d: \|x\| \leq n\}$. Set $a_n \equiv a\Phi_n$ and $\mu_n \equiv (0, \mu)\Phi_n$. Thanks to [9, Proposition IV.6.2], a_n has a globally Lipschitz continuous root. Moreover, μ_n is globally Lipschitz continuous as a continuously differentiable function with bounded derivative. Hence, [13, Theorem 5.2.5, Theorem 5.2.9, Proposition 5.3.20, Corollary 5.4.8] together with similar arguments as used in the proof of [13, Proposition 5.4.11] yield that the MPs (a_n, μ_n) are well-posed. \square

Remark 3.11. Let us comment on the incomplete case, i.e. $m < d$. Suppose that $a^{-1}b$ is locally bounded and that (3.9) holds. In this case Corollary 3.10 immediately implies that $|\mathcal{M}_l| = \infty$. Indeed, any locally bounded Borel function $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^{d-m}$ which satisfies the linear growth condition (3.10) gives rise to an ELMM.

We now give conditions for the existence of an EMM.

Condition E.1. For all $i = 1, \dots, m$ the following holds: There exists an $r > 0$ and continuous functions $A: [r, \infty) \rightarrow (0, \infty)$ and $B: [r, \infty) \rightarrow (0, \infty)$ such that for $\rho \geq \sqrt{2r}$ and $|x| = \rho$,

$$A\left(\frac{\rho^2}{2}\right) \geq \langle a(x)x, x \rangle,$$

$$\langle a(x)x, x \rangle B\left(\frac{\rho^2}{2}\right) \geq \text{trace } a(x) + 2\langle x, (0, \mu)(x) + a(x)e_i \rangle,$$

and (3.7) holds.

Condition E.2. For all $i = 1, \dots, m$ and $x \in \mathbb{R}^d$ there exists a constant $C > 0$ such that

$$\|a(x)\| \leq C(1 + \|x\|^2),$$

$$\langle x, (0, \mu)(x) + a(x)e_i \rangle \leq C(1 + \|x\|^2).$$

Similarly to the proof of Corollary 3.10, we deduce the following from Theorem 3.7.

Corollary 3.12. Suppose that (i) - (iii) from Corollary 3.10 hold. Moreover, suppose that

(iv) One of the Conditions E.1 and E.2 holds.

Then, $\emptyset \neq \mathcal{M}((0, \mu), a, x_0, T) \subseteq \mathcal{M}$. Moreover, if $m = d$, then $\mathcal{M}(0, a, x_0, T) = \mathcal{M}$ and $|\mathcal{M}|_{\mathcal{F}_T} = 1$.

We comment on classical Novikov-type conditions which are typically imposed to ensure the existence of an E(L)MM. Let c be as in Condition M.1. Then, it is well-known that an ELMM exists if

$$E^P \left[\exp \left(\frac{1}{2} \int_0^T \langle a(X_s)c_s, c_s \rangle ds \right) \right] < \infty.$$

This condition is simple to state. However, its verification is typically hard, if at all possible.

Let us give a simple example. Suppose that $m = d$ and that $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel function such that

$$\|b(x)\| \leq \text{const.} (1 + \|x\|), \quad x \in \mathbb{R}^d.$$

Moreover, let $a(x)$ be the unit matrix for all $x \in \mathbb{R}^d$. The MP (b, a) is well-posed, cf. [23, Theorem 10.2.2]. In other words, we suppose that P is the law of the solution process to the SDE

$$dY_t = b(Y_t) dt + dW_t, \quad Y_0 = x_0,$$

where W is a d -dimensional Brownian motion. Now, we can choose $c_s(\omega) \equiv -b(\omega(s))$ and Condition M.1 is satisfied due to the linear growth of b . Moreover, the Conditions U.1, EL.2 and E.2 hold and Corollary 3.12 yields $\mathcal{M} \neq \emptyset$. In this setup, Novikov's condition reads

$$(3.11) \quad E^P \left[\exp \left(\frac{1}{2} \int_0^T \|b(X_s)\|^2 ds \right) \right] < \infty.$$

For an easy example where (3.11) is violated consider

$$b(x_1, \dots, x_d) = (Cx_2, 0, \dots, 0),$$

for some large enough constant $C > 0$. Then, it holds that

$$E^P \left[\exp \left(\frac{1}{2} \int_0^T \|b(X_s)\|^2 ds \right) \right] = E \left[\exp \left(\frac{C^2}{2} \int_0^T \langle e_2, W_s \rangle^2 ds \right) \right] = \infty.$$

In this case, however, Novikov's condition can be relaxed using *salami tactics*, cf. [13, Corollary 3.5.14, Corollary 3.5.16].

Next, we give conditions for the non-existence of an E(L)MM in the complete case.

3.2.2. Deterministic Conditions for the Non-Existence of E(L)MMs. In this section we suppose that $m = d$. The following condition can be viewed as a complement to Condition EL.1. It is an extension of Khas'minskii's [14] criterion for explosion as given in [23, Theorem 10.2.4] and implies that the explosion happens arbitrarily fast.

Condition NL. 1. *There exists continuous functions $A: (0, \infty) \rightarrow (0, \infty)$ and $B: (0, \infty) \rightarrow (0, \infty)$ such that for $\rho > 0$ and $|x| = \rho$,*

$$(3.12) \quad A \left(\frac{\rho^2}{2} \right) \leq \langle a(x)x, x \rangle,$$

$$(3.13) \quad \langle a(x)x, x \rangle B \left(\frac{\rho^2}{2} \right) \leq \text{trace } a(x),$$

$$(3.14) \quad \int_{\frac{1}{2}}^{\infty} \frac{\int_{\frac{1}{2}}^z \frac{C(\sigma)}{A(\sigma)} d\sigma}{C(z)} dz < \infty,$$

$$(3.15) \quad \int_0^{\frac{1}{2}} \frac{\int_z^{\frac{1}{2}} \frac{C(\sigma)}{A(\sigma)} d\sigma}{C(z)} dz = \infty,$$

where

$$C(z) \equiv \exp \left(\int_{\frac{1}{2}}^z B(\sigma) d\sigma \right).$$

Moreover, for all $n \in \mathbb{N}$ there exist a strictly increasing function $\rho_n: [0, \infty) \rightarrow [0, \infty)$ with $\rho_n(0) = 0$ and a strictly increasing, concave and continuous function $\kappa_n: [0, \infty) \rightarrow [0, \infty)$ with $\kappa_n(0) = 0$ such that for all $x, y \in [\frac{1}{n}, n]$ it holds that

$$(3.16) \quad |A^{\frac{1}{2}}(x) - A^{\frac{1}{2}}(y)|^2 \leq \rho_n(|x - y|),$$

$$(3.17) \quad |A(x)B(x) - A(y)B(y)| \leq \kappa_n(|x - y|),$$

and for all $\epsilon > 0$

$$(3.18) \quad \int_0^{\epsilon} \frac{1}{\rho_n(z)} dz = \infty,$$

$$(3.19) \quad \int_0^{\epsilon} \frac{1}{\kappa_n(z)} dz = \infty.$$

Remark 3.13. The second part of Condition NL.1 is in the spirit of the Yamada-Watanabe [24] conditions for pathwise uniqueness. For instance, one may choose $\rho_n(x) = \kappa_n(x) = x$ if $A^{\frac{1}{2}}$ is locally Hölder continuous with exponent $\frac{1}{2}$ and AB is locally Lipschitz continuous.

The proof of the following theorem is based on a comparison argument together with the fact that one-dimensional diffusions explode (under appropriate conditions) arbitrarily fast.

Theorem 3.14. *Suppose that Condition NL.1 holds, then $\mathcal{M}_I = \emptyset$.*

Proof: In view of Theorem 3.5 it suffices to show that $\mathcal{M}(0, a, x_0, T) = \emptyset$. We use a contradiction argument. Suppose that $\mathcal{M}(0, a, x_0, T) \neq \emptyset$. Thanks to [13, Proposition 5.4.6, Proposition 5.4.11], there exists a filtered probability space which supports a Brownian motion W and a continuous adapted \mathbb{R}^d -valued process $(Y_t)_{t \in [0, T]}$ such that

$$dY_t = a^{\frac{1}{2}}(Y_t) dW_t, \quad Y_0 = x_0.$$

Define $p(x) \equiv \frac{1}{2}\|x\|^2$ and

$$\phi_t \equiv \inf \left(s \in [0, T] : \int_0^s \frac{\langle a(Y_r)Y_r, Y_r \rangle}{A(p(Y_s))} dr > t \right).$$

Then, using [21, Proposition V.1.4] and Itô's formula, we find a one-dimensional Brownian motion B such that

$$dp(Y_{\phi_t}) = A^{\frac{1}{2}}(p(Y_{\phi_t})) dB_t + \frac{1}{2} \frac{A(p(Y_{\phi_t})) \operatorname{trace} a(Y_{\phi_t})}{\langle a(Y_{\phi_t})Y_{\phi_t}, Y_{\phi_t} \rangle} dt, \quad p(Y_{\phi_0}) = \frac{1}{2}.$$

Let D_Δ be the one-point compactification of $D \equiv (0, \infty)$.

Lemma 3.15. *There exists an D_Δ -valued continuous process $(Z_t)_{t \in [0, \infty)}$ such that for all $n \in \mathbb{N}$ and $t \leq \gamma_n(Z)$*

$$(3.20) \quad dZ_t = A^{\frac{1}{2}}(Z_t) \mathbf{1}_{\{t \leq \gamma_n(Z)\}} dB_t + \frac{1}{2} A(Z_t) B(Z_t) \mathbf{1}_{\{t \leq \gamma_n(Z)\}} dt, \quad Z_0 = \frac{1}{2}.$$

Here, we set

$$\gamma_n(Z) \equiv \inf \left(t \in [0, \infty) : Z_t \notin \left(\frac{1}{n}, n \right) \right).$$

Proof: First, we prove a pathwise uniqueness result for the (stopped) SDE (3.20). Suppose that $(U_t)_{t \in [0, \infty)}$ and $(V_t)_{t \in [0, \infty)}$ are two continuous processes such that

$$\begin{aligned} dU_t &= A^{\frac{1}{2}}(U_t) \mathbf{1}_{\{t \leq \gamma_n(U)\}} dW_t + \frac{1}{2} A(U_t) B(U_t) \mathbf{1}_{\{t \leq \gamma_n(U)\}} dt, \quad U_0 = \frac{1}{2}, \\ dV_t &= A^{\frac{1}{2}}(V_t) \mathbf{1}_{\{t \leq \gamma_n(V)\}} dW_t + \frac{1}{2} A(V_t) B(V_t) \mathbf{1}_{\{t \leq \gamma_n(V)\}} dt, \quad V_0 = \frac{1}{2}, \end{aligned}$$

where W is a one-dimensional Brownian motion. Set

$$Y_t \equiv U_{t \wedge \gamma_n(U) \wedge \gamma_n(V)} - V_{t \wedge \gamma_n(U) \wedge \gamma_n(V)}.$$

Let ρ_n be as in Condition NL.1 and denote by $L^0(Y)$ the local time in 0 of the semimartingale Y . By (3.16) we have for all $t \in [0, \infty)$

$$\begin{aligned} (3.21) \quad \int_0^t \frac{1}{\rho_n(Y_s)} \mathbf{1}_{\{Y_s > 0\}} d\langle Y \rangle_s &= \int_0^t \frac{|A^{\frac{1}{2}}(U_s) - A^{\frac{1}{2}}(V_s)|^2}{\rho_n(|Y_s|)} \mathbf{1}_{\{Y_s > 0\}} \mathbf{1}_{\{s \leq \gamma_n(U) \wedge \gamma_n(V)\}} ds \\ &\leq \int_0^t \frac{\rho_n(|Y_s|)}{\rho_n(|Y_s|)} \mathbf{1}_{\{Y_s > 0\}} \mathbf{1}_{\{s \leq \gamma_n(U) \wedge \gamma_n(V)\}} ds \leq t, \end{aligned}$$

where $\langle \cdot \rangle$ denotes the quadratic variation process. Now, due to the hypothesis (3.18), [21, Lemma IX.3.3] yields that $L^0(Y) = 0$. Hence, by Tanaka's formula, cf., e.g., [21, Theorem VI.1.2], it holds that

$$|Y| = \int_0^\cdot \operatorname{sgn}(Y_s) dY_s.$$

Since,

$$(3.22) \quad \int_0^{t \wedge \gamma_n(U) \wedge \gamma_n(V)} \left(A^{\frac{1}{2}}(U_s) - A^{\frac{1}{2}}(V_s) \right)^2 ds \leq 2t \sup_{x \in [\frac{1}{n}, n]} |A(x)| < \infty,$$

the stochastic integral $\int_0^{\cdot \wedge \gamma_n(U) \wedge \gamma_n(V)} \text{sgn}(Y_s)(A^{\frac{1}{2}}(U_s) - A^{\frac{1}{2}}(V_s)) dW_s$ is a martingale. Hence, we obtain from (3.17), Fubini's theorem and Jensen's inequality

$$\begin{aligned} E[|Y_t|] &= \frac{1}{2} E \left[\int_0^{t \wedge \gamma_n(U) \wedge \gamma_n(V)} (A(U_s)B(U_s) - A(V_s)B(V_s)) ds \right] \\ &\leq \frac{1}{2} E \left[\int_0^t \kappa_n(|Y_s|) ds \right] \\ &= \frac{1}{2} \int_0^t E[\kappa_n(|Y_s|)] ds \\ &\leq \frac{1}{2} \int_0^t \kappa_n(E[|Y_s|]) ds. \end{aligned}$$

Now, due to the hypothesis (3.19), Bihari's lemma, cf. [1], yields $E[|Y_t|] = 0$. Hence, by continuity, a.s. $U_{t \wedge \gamma_n(U) \wedge \gamma_n(V)} = V_{t \wedge \gamma_n(U) \wedge \gamma_n(V)}$ for all $t \in [0, \infty)$. Moreover, thanks to [11, Lemma III.2.43], a.s. $\gamma_n(U) = \gamma_n(V)$ and we conclude that a.s. $U_{t \wedge \gamma_n(U)} = V_{t \wedge \gamma_n(V)}$ for all $t \in [0, \infty)$. In other words, we have shown that the SDE (3.20) satisfies pathwise uniqueness.

We now have to introduce some additional notation: Let Ω^Δ be the space of all continuous functions $\alpha: [0, \infty) \rightarrow D_\Delta$ such that $\alpha_s = \Delta$ for all $s \geq \gamma_\Delta(\alpha) \equiv \inf\{t \in [0, \infty): X_t = \Delta\}$. With abuse of notation denote by X the coordinate process on Ω^Δ and set $\mathcal{F}^\Delta \equiv \sigma(X_s, s \in [0, \infty))$, $\mathcal{F}_t^\Delta \equiv \sigma(X_s, s \in [0, t])$ and $\mathbf{F}^\Delta \equiv (\mathcal{F}_t^\Delta)_{t \in [0, \infty)}$. Moreover, let $\hat{\Omega}^\Delta$ be the space of all functions $\alpha: [0, \infty) \rightarrow D_\Delta$ which are continuous on $[0, \gamma_\Delta(\alpha))$ and $\alpha_s = \Delta$ for all $s \geq \gamma_\Delta$. Again, denote by X the coordinate process on $\hat{\Omega}^\Delta$ and set $\hat{\mathcal{F}}^\Delta \equiv \sigma(X_s, s \in [0, \infty))$, $\hat{\mathcal{F}}_t^\Delta \equiv \sigma(X_s, s \in [0, t])$ and $\hat{\mathbf{F}}^\Delta \equiv (\hat{\mathcal{F}}_t^\Delta)_{t \in [0, \infty)}$. For simplicity, we write γ_n instead of $\gamma_n(X)$.

Since $A^{\frac{1}{2}}$ and AB are continuous, [19, Theorem 1.13.1] yields that there exists a probability measure Q on $(\Omega^\Delta, \mathcal{F}^\Delta)$ such that for all $f \in C_c^2(D)$ and all $n \in \mathbb{N}$ the process

$$f(X_{\cdot \wedge \gamma_n}) - \int_0^{\cdot \wedge \gamma_n} \left(\frac{1}{2} A(X_s)B(X_s)f'(X_s) + A(X_s)f''(X_s) \right) ds$$

is an (\mathbf{F}^Δ, Q) -martingale. Note that $Q \circ X_{\cdot \wedge \gamma_n}^{-1} = Q$ on $\mathcal{F}_{\gamma_n}^\Delta$ and $\gamma_n = \gamma_n(X_{\cdot \wedge \gamma_n})$, cf., e.g., [22, Section 1.2]. Hence, by [13, Corollary 5.4.8, Proposition 5.4.11], for all $n \in \mathbb{N}$ there exists a weak solution to the SDE (3.20) which law coincides with Q on $\mathcal{F}_{\gamma_n}^\Delta$. Since we already established that the SDE (3.20) satisfies pathwise uniqueness, the fact that weak existence together with pathwise uniqueness implies strong existence, cf., e.g., [21, Theorem IX.17], implies that there exists a continuous process Z^n such that

$$dZ_t^n = A^{\frac{1}{2}}(Z_t^n) \mathbf{1}_{\{t \leq \gamma_n(Z^n)\}} dB_t + \frac{1}{2} A(Z_t^n)B(Z_t^n) \mathbf{1}_{\{t \leq \gamma_n(Z^n)\}} dt, \quad Z_t^n = \frac{1}{2}.$$

By the Yamada-Watanabe theorem, cf., e.g., [11, Theorem IX.1.7], pathwise uniqueness also yields that the law of Z^n coincides with Q on $\mathcal{F}_{\gamma_n}^\Delta$. In particular, a.s. $Z_{\cdot \wedge \gamma_n(Z^{n+1})}^{n+1} = Z_{\cdot \wedge \gamma_n(Z^n)}^n$ and a.s. $\gamma_n(Z^{n+1}) = \gamma_n(Z^n)$, cf. again, e.g., [22, Section 1.2]. Set

$$Z_t \equiv \begin{cases} Z_t^n & \text{if } t \leq \gamma_n(Z^n), \\ \Delta & \text{otherwise.} \end{cases}$$

It remains to prove that Z has continuous paths. Clearly, Z has paths in $\hat{\Omega}^\Delta$. Moreover, the law of Z coincides with Q on $\hat{\mathcal{F}}_{\gamma_n}^\Delta$ for all $n \in \mathbb{N}$. By Parthasarathy's extension theorem, cf. [18], there exists a unique probability measure Q^Δ on $(\hat{\Omega}^\Delta, \hat{\mathcal{F}}^\Delta)$ such that Q^Δ equals the law of Z on $\hat{\mathcal{F}}_{\gamma_n}^\Delta$ for all $n \in \mathbb{N}$. Hence, Q^Δ is necessarily

the law of Z . In particular, the uniqueness yields that Q^Δ equals Q when extended to $(\hat{\Omega}^\Delta, \hat{\mathcal{F}}^\Delta)$ in the obvious manner. Therefore, Z is continuous and the proof is finished.

Let us verify the assumptions of Parthasarathy's extension theorem. We have to show the following:

- (i) For all $n \in \mathbb{N}$ the space $(\hat{\Omega}^\Delta, \hat{\mathcal{F}}_{\gamma_n}^\Delta)$ is a standard Borel space.
- (ii) $\bigvee_{n \in \mathbb{N}} \hat{\mathcal{F}}_{\gamma_n}^\Delta = \hat{\mathcal{F}}^\Delta$.
- (iii) If $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ is an increasing sequence and $(A_k)_{k \in \mathbb{N}}$ is a decreasing sequence such that $A_k \in \hat{\mathcal{F}}_{\gamma_{n_k}}^\Delta$ is an atom, it holds that $\bigcap_{k \in \mathbb{N}} A_k \neq \emptyset$.

Note that $\hat{\Omega}^\Delta$ is a closed subspace of the set of continuous functions $[0, \infty) \rightarrow D_\Delta$, which is a Polish space, cf. [19, p. 40]. Hence, $\hat{\Omega}^\Delta$ is a Polish space itself. Moreover, $\hat{\mathcal{F}}^\Delta$ is its Borel σ -field, cf. again [19, p. 40]. Since $\hat{\mathcal{F}}_{\gamma_n}^\Delta = \sigma(X_{t \wedge \gamma_n}, t \in [0, \infty))$, cf. [22, Section 1.3], the σ -field $\hat{\mathcal{F}}_{\gamma_n}^\Delta$ is countably generated. Hence, (i) follows from [18, Theorem V.2.4]. Part (ii) is well-known, cf. again [19, p. 40]. Finally, (iii) follows as in [7, Remark 6.1]. \square

Next, we compare Z and $U \equiv p(Y_\phi)$. We set

$$\xi_n \equiv \inf \left(t \in [0, T] : U_t \notin \left(\frac{1}{n}, n \right) \right).$$

Here, we use that $\phi_t \leq t$.

Lemma 3.16. *For all $n \in \mathbb{N}$ a.s.*

$$(3.23) \quad Z_{\cdot \wedge \gamma_n(Z) \wedge \xi_n \wedge T} \leq U_{\cdot \wedge \gamma_n(Z) \wedge \xi_n \wedge T}.$$

Proof: We set $Y^n \equiv Z_{\cdot \wedge \gamma_n(Z) \wedge \xi_n \wedge T} - U_{\cdot \wedge \gamma_n(Z) \wedge \xi_n \wedge T}$. Let $t \in [0, \infty)$ be fixed. As in (3.21), we obtain

$$\int_0^t \frac{1}{\rho_n(Y_s^n)} \mathbf{1}_{\{Y_s^n > 0\}} d\langle Y^n \rangle_s \leq t.$$

Hence, by [21, Lemma IX.3.3], $L^0(Y^n) = 0$ and by Tanaka's formula

$$\max(Y_t^n, 0) = \int_0^t \mathbf{1}_{\{Y_s^n > 0\}} dY_s^n.$$

Similar to (3.22) one verifies that the Brownian part of Y^n is a martingale. Now, Fubini's theorem and Jensen's inequality yield

$$\begin{aligned} & E[\max(Y_t^n, 0)] \\ &= E \left[\int_0^{t \wedge \gamma_n(Z) \wedge \xi_n \wedge T} \mathbf{1}_{\{Z_s > U_s\}} \frac{1}{2} \left(A(Z_s)B(Z_s) - A(U_s) \frac{\text{trace } a(Y_{\phi_s})}{\langle a(Y_{\phi_s})Y_{\phi_s}, Y_{\phi_s} \rangle} \right) ds \right] \\ &\leq E \left[\int_0^{t \wedge \gamma_n(Z) \wedge \xi_n \wedge T} \mathbf{1}_{\{Z_s > U_s\}} \frac{1}{2} (A(Z_s)B(Z_s) - A(U_s)B(U_s)) ds \right] \\ &\leq \frac{1}{2} \int_0^t \kappa_n(E[\max(Y_s^n, 0)]) ds. \end{aligned}$$

By Bihari's lemma, $E[\max(Y_t^n, 0)] = 0$. Using the continuity of Y^n , this implies the claim. \square

The previous lemma states that a.s. the paths of Z are below the paths of U till either Z or U leave $(0, \infty)$ or time T . However, by the Feller test for explosion, cf. [19, Theorem 5.1.5], the condition (3.13) yields that Z a.s. does not explode to

0. Hence, by the pathwise ordering of Z and U , also U a.s. does not explode to 0. By the Feller test and [12, Theorem 4.8], (3.12) implies that Z explodes to $+\infty$ arbitrarily fast, i.e. before time T with positive probability. Therefore, our pathwise ordering implies that U explodes to $+\infty$ with positive probability. We deduce from $\phi_t \leq t$ that $\|Y\|$ explodes to $+\infty$ with positive probability. This, however, is a contradiction to the fact that Y is an \mathbb{R}^d -valued process. Hence, we conclude that $\mathcal{M}(0, a, x_0, T) = \emptyset$. This finishes the proof. \square

In the same manner we can formulate a condition for the existence of an ELMM which is no EMM.

Condition N.1. *For at least one $i \in \{1, \dots, d\}$ the following holds: There exists continuous functions $A: (0, \infty) \rightarrow (0, \infty)$ and $B: (0, \infty) \rightarrow (0, \infty)$ such that for $\rho > 0$ and $|x| = \rho$,*

$$(3.24) \quad \begin{aligned} A\left(\frac{\rho^2}{2}\right) &\leq \langle a(x)x, x \rangle, \\ \langle a(x)x, x \rangle B\left(\frac{\rho^2}{2}\right) &\leq \text{trace } a(x) + 2\langle x, a(x)e_i \rangle, \end{aligned}$$

and (3.14) and (3.15) hold. Moreover, for all $n \in \mathbb{N}$ there exist a strictly increasing function $\rho_n: [0, \infty) \rightarrow [0, \infty)$ with $\rho_n(0) = 0$ and a strictly increasing, concave and continuous function $\kappa_n: [0, \infty) \rightarrow [0, \infty)$ with $\kappa_n(0) = 0$ such that for all $x, y \in [\frac{1}{n}, n]$ it holds that

$$(3.25) \quad |A^{\frac{1}{2}}(x) - A^{\frac{1}{2}}(y)|^2 \leq \rho_n(|x - y|),$$

$$(3.26) \quad |A(x)B(x) - A(y)B(y)| \leq \kappa_n(|x - y|),$$

and for all $\epsilon > 0$

$$(3.27) \quad \int_0^\epsilon \frac{1}{\rho_n(z)} dz = \infty,$$

$$(3.28) \quad \int_0^\epsilon \frac{1}{\kappa_n(z)} dz = \infty.$$

The following theorem can be shown similarly to Theorem 3.14. We omit the details.

Theorem 3.17. *If Condition N.1 holds, then $\mathcal{M} = \emptyset$.*

Remark 3.18. If the conditions of Corollary 3.10 hold and Condition N.1 is satisfied, then there exists a (unique on \mathcal{F}_T) ELMM and no EMM. In this case the market includes a financial bubble as defined by Cox and Hobson [8].

4. APPLICATION: THE INFLUENCE OF THE MARKET DIMENSION

The number of assets in a market plays a crucial role for the existence of an ELMM. In this section we give an example for a financial market in which the number of risky assets coincides with the number of the sources of risk and the existence of an ELMM depends on the number of assets.

Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function such that for all $R > 0$

$$\inf_{\|x\| \leq R} f(x) > 0.$$

We suppose that $m = d$ and that

$$\langle a(x)e_i, e_j \rangle \equiv f(x)\mathbf{1}_{\{i=j\}}.$$

In this setup, Condition M.1 is satisfied with $c_s(\omega) = a^{-1}(\omega(s))b(\omega(s))$. We transfer Condition NL.1 in this setting.

Condition NL.2. Assume that there exists a locally Lipschitz continuous function $\alpha: (0, \infty) \rightarrow (0, \infty)$ such that $\alpha(\rho) \leq f(x)$ for $\rho > 0$ and $|x| = \rho$, and

$$(4.1) \quad \int_1^\infty \frac{1}{\alpha(\sqrt{\rho})} d\rho < \infty.$$

Corollary 4.1. If $d \leq 2$, then $\mathcal{M}_l \neq \emptyset$ and $|\mathcal{M}_l|_{\mathcal{F}_T} = 1$. If $d \geq 3$ and Condition NL.2 holds, then $\mathcal{M}_l = \emptyset$.

Proof: If $d \leq 2$, then the MP $(0, a, x_0)$ is well-posed, cf. [23, Exercise 10.3.3]. Hence, the claims follow from Corollary 3.6.

Suppose now that $d \geq 3$ and that Condition NL.2 holds. Set $A(x) = 2x\alpha(\sqrt{2x})$ and $B(x) = \frac{d}{2x}$ for $x \in (0, \infty)$. It is routine to check that A and B satisfy (3.12) and (3.13). Moreover, since compositions and products of locally Lipschitz continuous functions are locally Lipschitz continuous, $A^{\frac{1}{2}}$ and AB are locally Lipschitz continuous. Let C be defined as in (3.8). By Fubini's theorem, using that $d \geq 3$, we obtain

$$\begin{aligned} \int_{\frac{1}{2}}^\infty \frac{\int_{\frac{1}{2}}^z \frac{C(\sigma)}{A(\sigma)} d\sigma}{C(z)} dz &= \int_{\frac{1}{2}}^\infty \int_\sigma^\infty \frac{1}{C(z)} dz \frac{C(\sigma)}{A(\sigma)} d\sigma \\ &= \text{const.} \int_{\frac{1}{2}}^\infty \frac{\sigma}{A(\sigma)} d\sigma \\ &= \text{const.} \int_{\frac{1}{2}}^\infty \frac{1}{\alpha(\sqrt{2\sigma})} d\sigma. \end{aligned}$$

Hence, (4.1) is equivalent to (3.14). Moreover, using again $d \geq 3$, we obtain

$$\int_0^{1/2} \exp\left(-\int_{1/2}^z B(u) du\right) dz = \text{const.} \int_0^{1/2} \frac{1}{z^{d/2}} dz = +\infty.$$

Hence, [13, Problem 5.5.27] implies that (3.15) is satisfied. Putting these pieces together, Condition NL.2 implies Condition NL.1. Therefore, we conclude $\mathcal{M}_l = \emptyset$ from Theorem 3.14. \square

In other words, Corollary 4.1 states that in the one and two asset case there always exists an ELMM. However, there are situations with more than three assets where no ELMM exists. This illustrates the existence of financial markets including arbitrage opportunities where the number of assets coincides with the number of sources of risk.

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D. CRIENS - TECHNICAL UNIVERSITY OF MUNICH, DEPARTMENT OF MATHEMATICS, GERMANY
 E-mail address: david.criens@tum.de